Integrable lattice models from susy gauge theories

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There are several connections of integrable models to supersymmetric gauge theories

[Spiridonov, Kels, Yagi, Razamat ...]

One of such connections is a correspondence between integrable lattice models and supersymmetric quiver gauge theories such that the integrability emerges as a manifestation of supersymmetric duality.

The idea of the correspondence allows one to obtain solutions to the Yang-Baxter equation

Exact results in SUSY

The full information about a local quantum field theory is encoded in the euclidean path-integral

$$Z = \int D\phi \, e^{-S}$$

over all possible field configurations in Euclidean spacetime.

In general: Hard to compute.

Localization: Allows to exactly compute the partition function

- used in cohomological and topological field theories
- recently: 4d $\mathcal{N}=2$ th.: Ω- background by Nekrasov
  S^4 by Pestun

Consider SUSY theory on a manifold with a supercharge Q

Partition function

$$Z(t) = \int D\phi \, e^{-(S + t\{Q, V\})}$$

positive semi-definite
Q-invariant functional

In fact Z is independent of t : $\frac{dZ}{dt} = 0$

For large t the path integral only gets contributions near $QV_{bos} = 0$

Matrix integral : $Z \sim \oint \prod^{\mathrm{rank}\, G} Z_{gauge} \prod^{\mathrm{rank}\, F} Z_{chiral}$

[talk by Razamat]

Hypergeometric integrals

Partition function on

$S^3 \times S^1$, $S^3/\mathbb{Z}_r \times S^1$: elliptic [Spiridonov, Razamat, Brünner, ...]

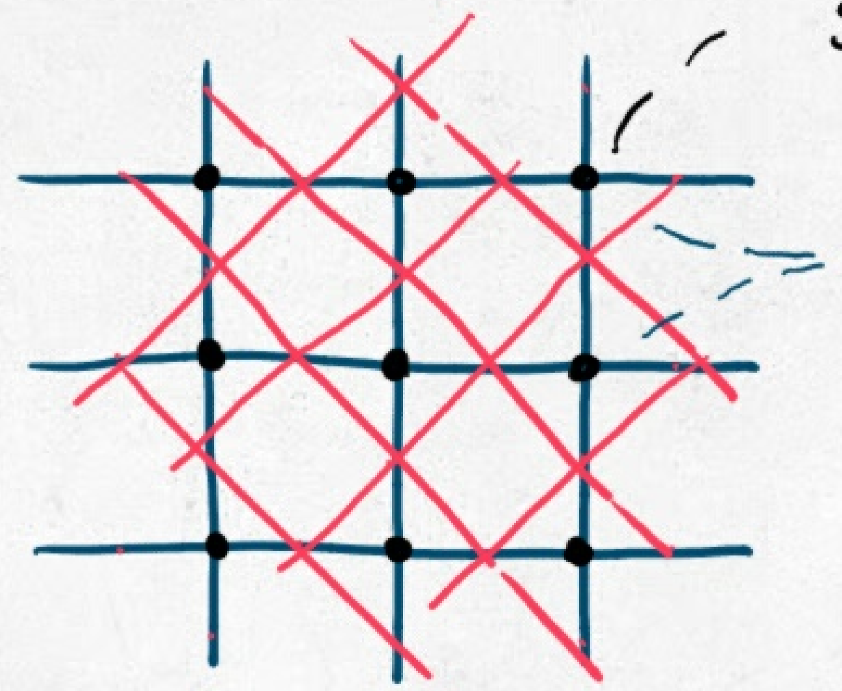
$S^2 \times S^1$: basic [Spiridonov, Razamat, Krattenthaler, Rosengren, ...]

S^3_b, $S^3/\mathbb{Z}_r$: hyperbolic [Spiridonov, Koroteev, ...]

S^2 : ordinary

A crash course on integrable lattice models

rapidity lines



Spin : $\sigma_i = (x_i, m_i)$ $\quad x_i \in \mathbb{R}$, $m_i \in \mathbb{Z}$

two spins σ_i, σ_j interact if they connected by edge (ij)

Boltzmann weights

$\mathcal{W}_{pq}(\sigma_i, \sigma_j)$ $\qquad \overline{\mathcal{W}}_{pq}(\sigma_i, \sigma_j)$

- $\mathcal{W}_\alpha = \mathcal{W}_{p-q}$
- $\overline{\mathcal{W}}_\alpha = \mathcal{W}_{\eta-\alpha}$
- $\mathcal{W}(\sigma_i, \sigma_j) = \mathcal{W}(\sigma_j, \sigma_i)$

The partition function

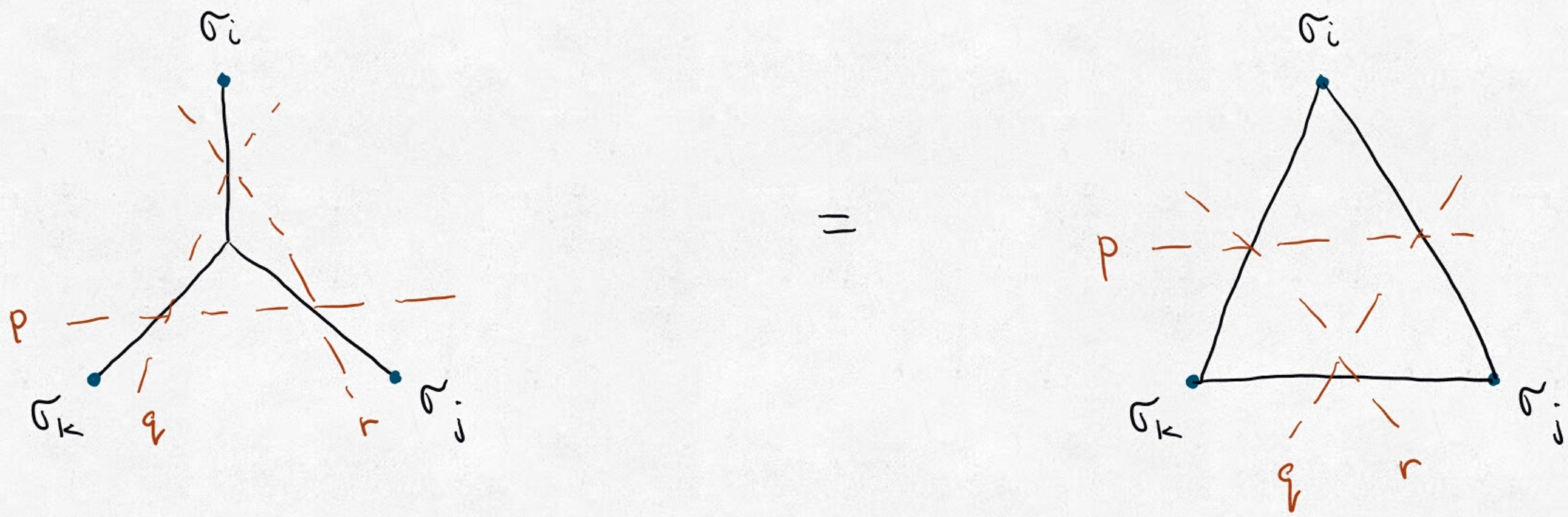
$$Z = \oint\!\!\!\!\!\!\!\sum \prod_{(ij)} \mathcal{W}_{\alpha_{ij}}(\sigma_i, \sigma_j) \prod_{(kl)} \mathcal{W}_{\eta - \alpha_{kl}}(\sigma_k, \sigma_l) \prod_n S(\sigma_n)$$

run over all possible values of internal spins

The model is **integrable** if one can evaluate the partition function in the thermodinamic limit $N \to \infty$.

Star-triangle relation

An exact evaluation is possible if the Boltzmann weights satisfy the Yang-Baxter equation, which for the models we consider here takes the form of the following star-triangle relation



$$\sum\!\!\!\!\!\!\int S(\sigma)\, W_{\eta-\alpha_i}(\sigma_i,\sigma)\, W_{\eta-\alpha_j}(\sigma_j,\sigma)\, W_{\eta-\alpha_k}(\sigma_k,\sigma) = R(\alpha_i,\alpha_j,\alpha_k)\, W_{\alpha_i}(\sigma_j,\sigma_k)\, W_{\alpha_j}(\sigma_i,\sigma_k)\, W_{\alpha_k}(\sigma_j,\sigma_i)$$

Seiberg duality

Supersymmetric theories with four supercharge

Electric theory : $SU(2)$ gauge group and quark superfields in the fundamental representation of the $SU(6)$ flavor group

Magnetic theory : no gauge group, the matter sector contains meson superfields in 15-dimensional antisymmetric $SU(6)$-tensor representation of the second rank

Duality has different features in different dimensions, but such details are not crucial for our discussion.

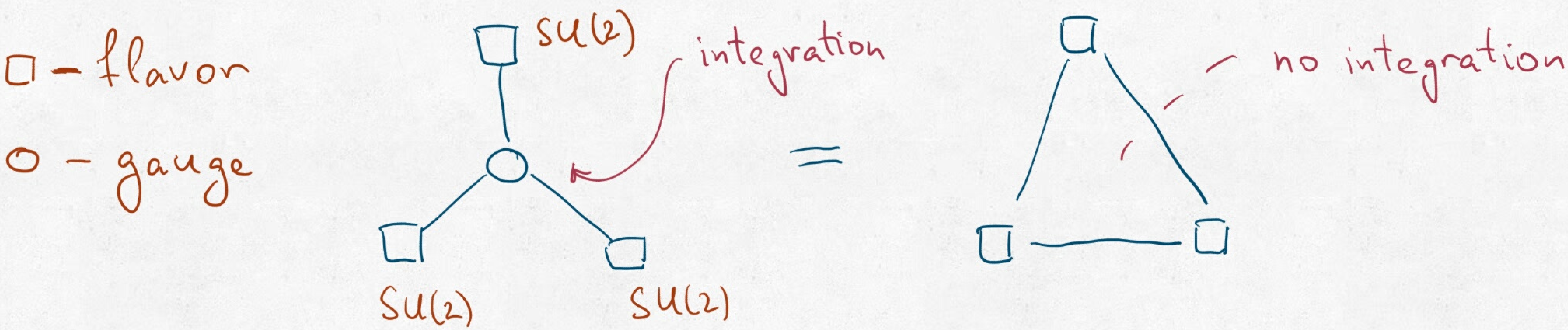
Partition functions

4d supersymmetric index : elliptic beta integral [Spiridonov]

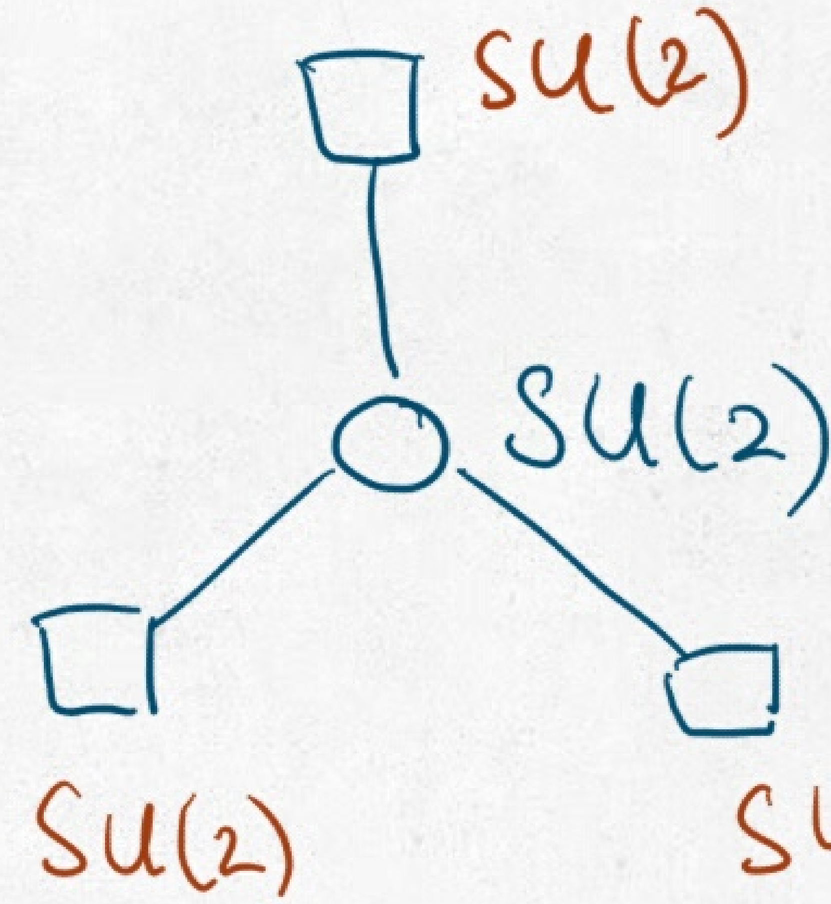
3d supersymmetric index : q-beta sum/integral [Rosengren]

3d sphere partition func. : hyperbolic beta integral [Bult, Rains, Stokman]

By adding a certain superpotential one may break flavor symmetry of both theories from $SU(6)$ down to $SU(2) \times SU(2) \times SU(2)$



Relation to integrable models



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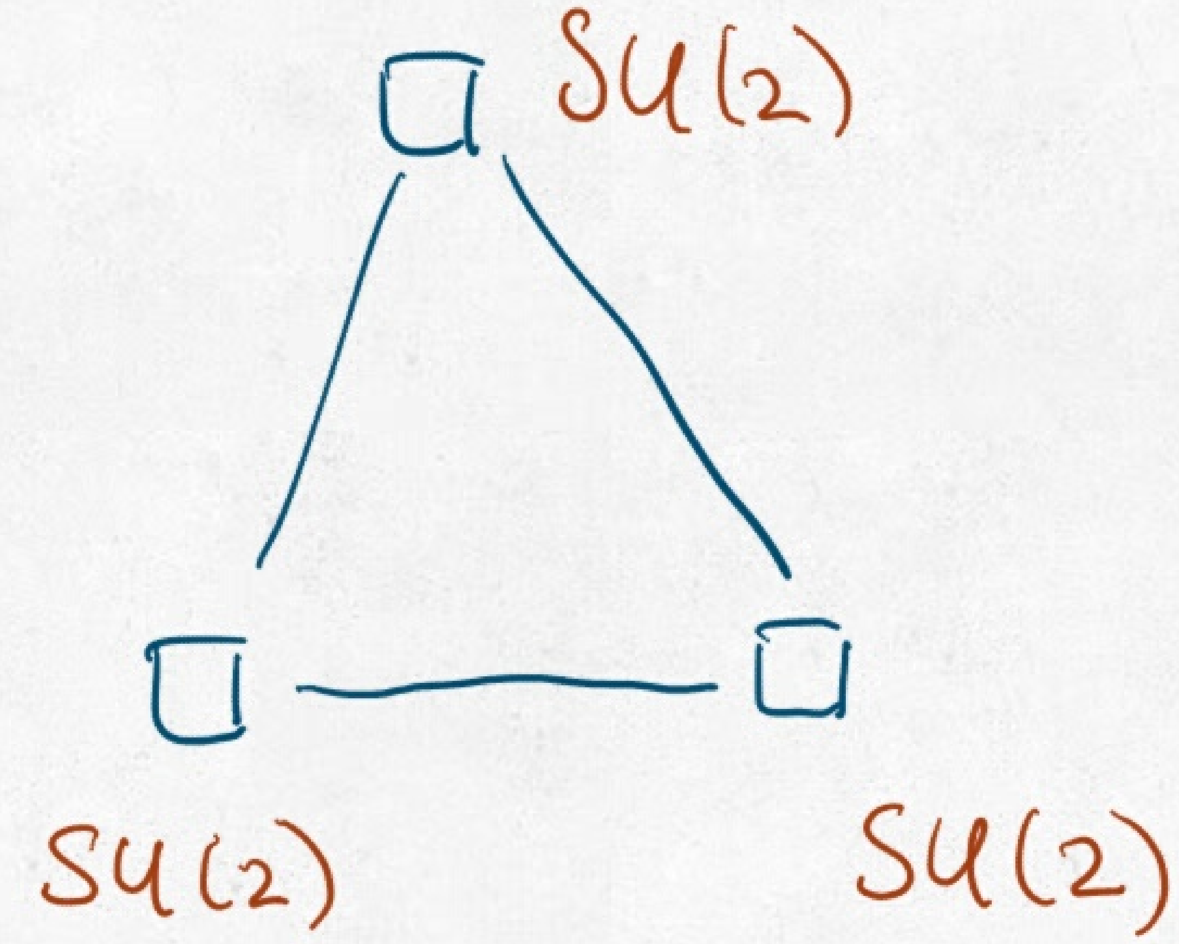


Star-triangle relation $\longleftrightarrow$ Seiberg duality

$$\oint S(\sigma)\, W_{\eta-\alpha_i}(\sigma_i,\sigma)\, W_{\eta-\alpha_j}(\sigma_j,\sigma)\, W_{\eta-\alpha_k}(\sigma_k,\sigma)$$

$$= R(\alpha_i,\alpha_j,\alpha_k)\, W_{\alpha_i}(\sigma_j,\sigma_k)\, W_{\alpha_j}(\sigma_i,\sigma_k)\, W_{\alpha_k}(\sigma_j,\sigma_i)$$

fugacity for gauge group

contribution of vector multiplet

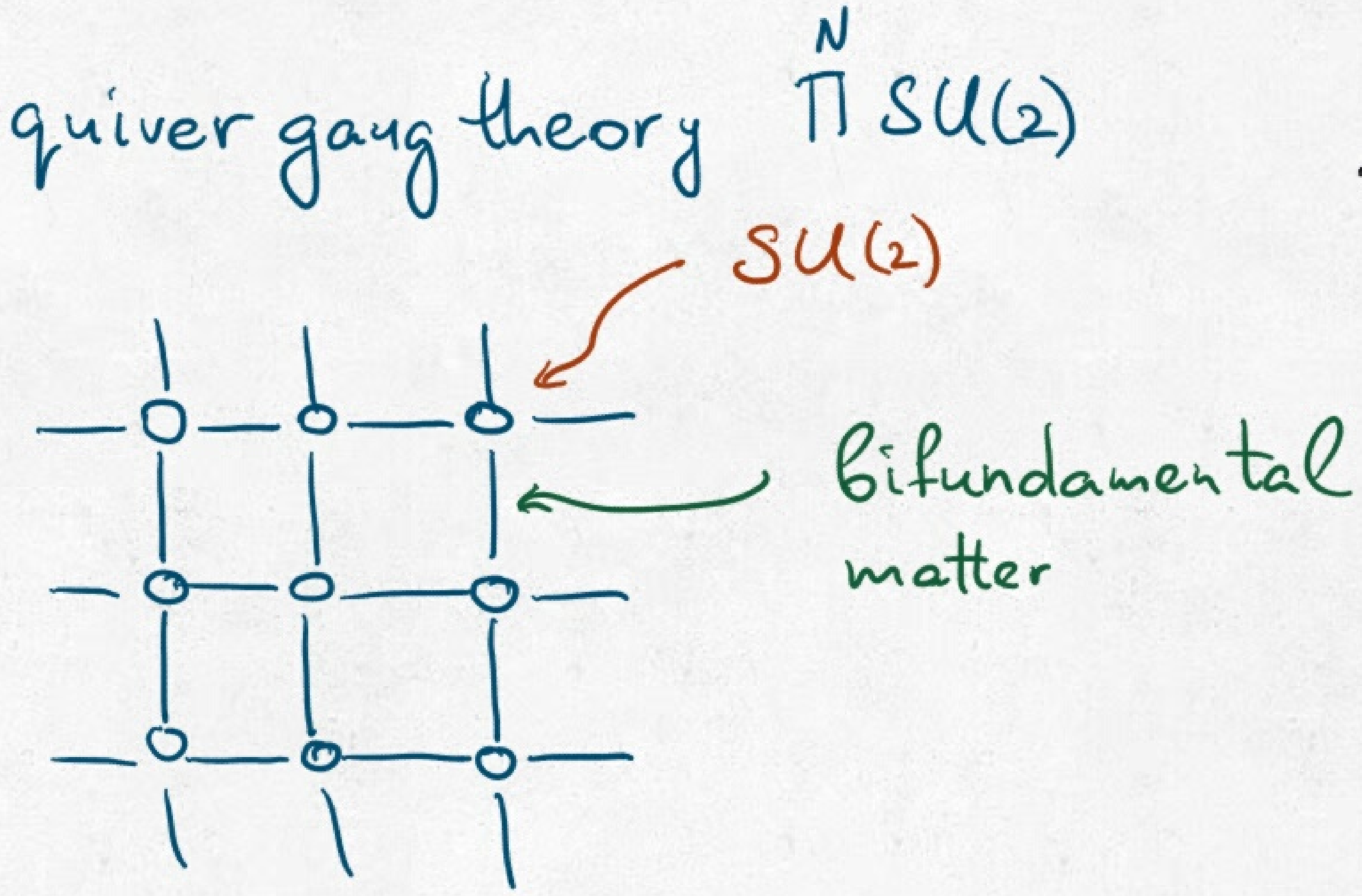
contribution of chiral multiplet

R-charge

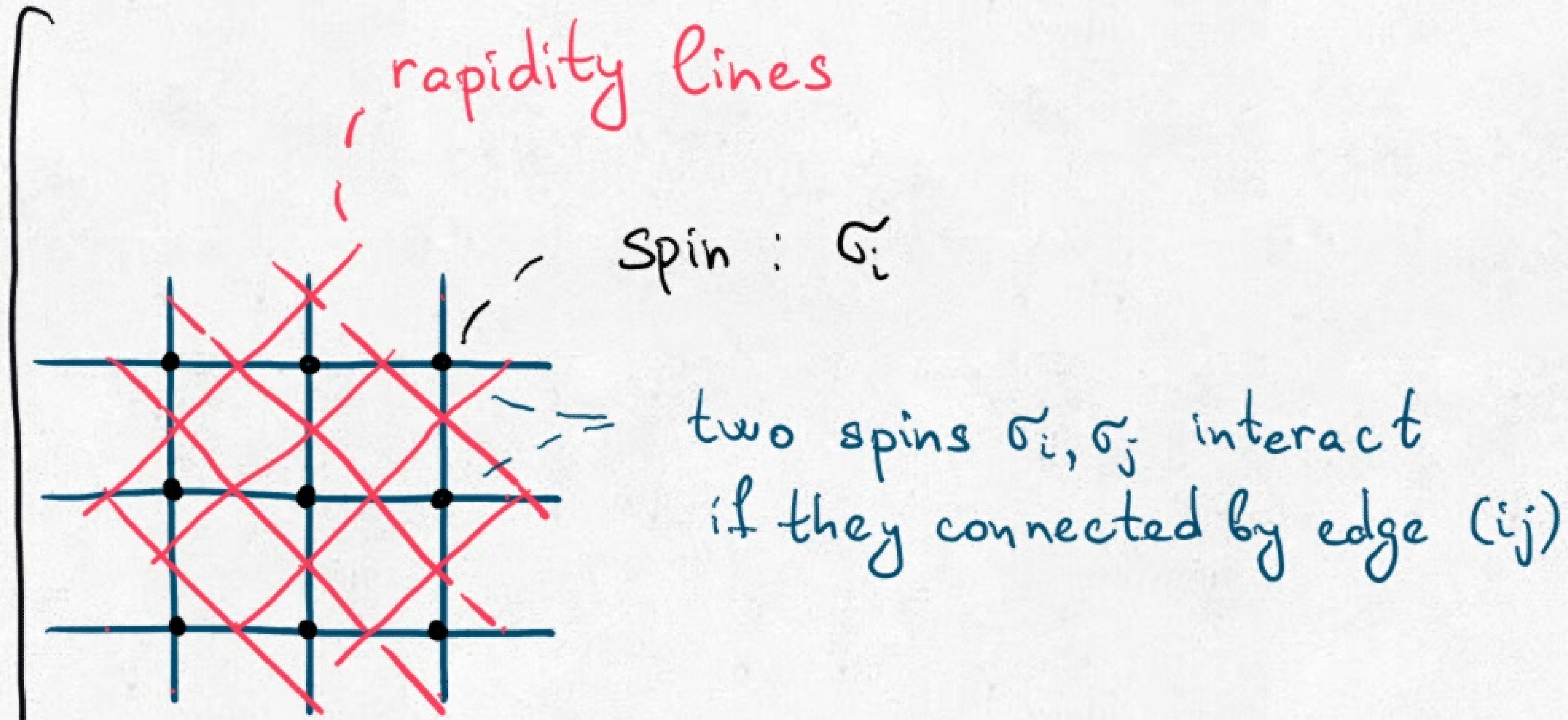
fugacities for flavor group

[Spiridonov] [Yamazaki] ...

Z [



] $= Z$ [



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- Inversion relation satisfied by Boltzmann weights corresponds to the chiral symmetry breaking of the corresponding supersymmetric gauge theory.

Brane construction : [Yagi]

Solutions of YB eq.

Partition function on $S^3/\mathbb{Z}$

Lens space: $S^3_b = \{(x,y) \in \mathbb{C}^2 \mid b^2|x|^2 + b^{-2}|y|^2 = 1\}$ with $(x,y) \sim (e^{\frac{2\pi i}{r}} x, e^{-\frac{2\pi i}{r}} y)$

Spins: $\sigma_i = (x_i, m_i) \quad 0 \le x_i < \infty, \quad m_i = 0, \ldots, r-1$

Boltzmann weights:

$$W_\alpha(\sigma_i, \sigma_j) = \frac{1}{\chi(\alpha)} \frac{\varphi_{m_i \pm m_j}(x_i \pm x_j + i\alpha)}{\varphi_{m_i \pm m_j}(x_i \pm x_j - i\alpha)}$$

$$S(\sigma_i) = \frac{1}{r\sqrt{\omega_1 \omega_2}} \varphi_{\pm 2m_j}(\pm 2x_j - i\eta)$$

[IG, Kels]

Here

$$\varphi_{r,m}(z) = \exp\left[\int_0^\infty \frac{dx}{x}\left(\frac{iz}{\omega_1\omega_2 r x} - \frac{\sinh(2izx - \omega_1(r-2[m]x)}{2\sinh(\omega_1 r x)\sinh(2\eta x)} - \frac{\sinh(2izx + \omega_2(r-2[m]x)}{2\sinh(\omega_2 r x)\sinh(2\eta x)}\right)\right]$$

Sum/integral extension of hyperbolic beta integral.

$\downarrow$

new integral identities similar to identities from Bult's thesis

[Nieri, Pasquetti]

Can be obtained from lens elliptic hypergeometric integrals

[Kels, Spiridonov, Yamazaki, ...]

Star-star relation $\longleftrightarrow$ integral with $W(E_7)$ symmetry

Solutions of YB eq.

Supersymmetric index $(S^2 \times S^1)$:

$$\text{Spin:}\quad \sigma_i = (\underset{\mathbb{R}}{x_i}, \underset{\mathbb{Z}}{m_i})$$

$$W_\alpha(\sigma_i, \sigma_k) = \frac{q^{-2i(x_i m_i + x_k m_k)}}{\underset{\exp\left(-\sum_{n=1}^{\infty} \frac{e^{4\alpha n}}{n(q^n - q^{-n})}\right)}{\underbrace{\kappa(\alpha)}}} \frac{\left(q^{1+\frac{m_i \pm m_k}{2} + \eta - \alpha - i(x_i \pm x_k)}; q\right)_\infty}{\left(q^{\frac{m_i \pm m_k}{2} + \alpha - \eta + i(x_i \pm x_k)}; q\right)_\infty}$$

usual q-Pochhammer

$$S(\sigma_0) = \frac{1}{q^m} \frac{\left(q^{\pm 2x_0 + m}; q\right)_\infty}{\left(q^{\pm 2x_0 + m + 1}; q\right)_\infty}$$

[JG, Spiridonov] [Kels]

Star-triangle $\longleftarrow$ Rosengren's q-beta sum/integral

Boltzmann weights with $W(x,y) \neq W(y,x)$

2d vortex and anti-vortex PF

$$\int S(z)\, \overline{W}_\alpha(x,z)\, W_\gamma(w,z)\, \overline{W}_\beta(z,y) = R_{\alpha\beta\gamma}\, W_\alpha(w,y)\, \overline{W}_\gamma(x,y)\, W_\beta(w,x)$$

$$\int S(z)\, \overline{W}_\alpha(z,x)\, W_\gamma(z,w)\, \overline{W}_\beta(y,z) = R_{\alpha\beta\gamma}\, W_\alpha(y,w)\, \overline{W}_\gamma(y,x)\, W_\beta(x,w)$$

! Not physical

$$\overline{W}_\alpha(x,z) = \Gamma(\alpha \pm ix \pm iz) \qquad W_\alpha(x,z) = \frac{\Gamma(-\alpha + ix \pm iz)}{\Gamma(\alpha + ix \pm iz)}$$

$$S(z) = \frac{1}{\Gamma(\pm 2iz)} \qquad R_{\alpha\beta\gamma} = \frac{\Gamma(2\alpha)\,\Gamma(2\beta)}{\Gamma(2\gamma)}$$

? Extension to 3d hemi-sphere PF, other dimensions

Relation to the Painleve

$$\text{const} \int \frac{dz}{2\pi i z} \prod_{i=1}^{8} \frac{\Gamma(t_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)}$$ $\leftarrow$ τ-function of the discrete Painleve system of type $E_8^{(1)}$

[Spiridonov]

[Noumi]

Painlevé equations are at the borderline between trivial integrability and nonintegrability.

$\uparrow$

R-matrix of the IRF version of Bazhanov-Sergeev lattice spin model

[Chicherin]

Quantum integrability

Other solutions $\longleftrightarrow$?

Summary and comments

- New solutions to the Yang-Baxter equation in terms of hypergeometric functions

- Integrable models corresponding to other Seiberg dualities ?
- Origin of solutions in the framework of the representation theory of quantum group [Yagi]

- Extension to the tetrahedron equation by Zamolodchikov
- New sum/integral identities

Thank you!